Graphs of fixed order and size with maximal A_α-index

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Abstract

For any real number $\alpha \in [0,1]$, by the A_α-*matrix* of a graph G we mean the matrix $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-index of G. We settle the problem of characterizing graphs which attain the maximum A_α-index over $\mathcal{G}(n, n+k)$, the class of graphs with n vertices and $n+k$ edges, for $-1 \leq k \leq n-3$ and $\frac{1}{2} \leq \alpha < 1$. The following result is obtained: for $-1 \leq k \leq n-3$, when $\frac{1}{2} \leq \alpha < 1$, $H_{n,k}$ is the unique graph in $\mathcal{G}(n, n+k)$ that maximizes the A_α-index, except when $(n,k) = (4,-1), (n,2)$ or $(7,3)$ and $\alpha = \frac{1}{2}$, or $(n,k) = (5,1)$ and $\alpha \in [\frac{1}{2}, \frac{35-\sqrt{409}}{24}]$. Our work completes the corresponding work of Chang and Tam (2010) and Zhai et al.(2022) for the special case $\alpha = \frac{1}{2}$. As a by-product, we provide a new proof for the known result that for any positive integer m and any real number $\alpha \in [\frac{1}{2}, 1)$, if $(m,\alpha) \neq (3, \frac{1}{2})$, then a graph maximizes the A_α-index over all graphs with m edges if and only if it is the union of $K_{1,m}$ with a (possibly empty) null graph; a graph maximizes the $A_{\frac{1}{2}}$-index over all graphs with three edges if and only if it is the union of $K_{1,3}$ or K_3 with a (possibly empty) null graph. Some open questions are also posed.

Keywords: Maximal A_α-index problem; Maximal graph; Threshold graph; Neighborhood equivalence classes; Quasi-complete graphs; Quasi-stars

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